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An Investigation of the Best Choice Problem in Posets Through Colored Binary Tree Structures

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Abstract

We look at the poset version of the secretary issue for rooted complete binary trees of length n, where the 2na complete binary trees with roots at level a+1 (counting from the leaves) are colored with various hues that are visible to the eye.

The selector and the vertices above level a + 1 are naturally colored based on the vertices that came before them. We discover a near-optimal strategy for more than two colors as well as an optimal halting time for trees with only two colors.

1 Introduction

The secretary problem is a well-known name for the following issue. There are n linearly arranged things. A selection goes through each one in a random permutation. Only the objects that have already undergone examination by the selector can be compared to the current one. The selector's goal is to select the current item with the highest likelihood that it is the best option available. The term "secretary problem" refers to a fun variation of the problem where a selector (the administrator) evaluates applicants (our linearly ordered objects) for the position of a secretary with the objective of selecting the candidate with the highest probability online. For a solution to this issue, see Lindley (1961).

There has been a lot of interest in this issue. It was given consideration in several enriched forms. Ferguson's study is intriguing (1989). The secretary problem naturally generalizes to posets as well. Specifically, we can suppose that the selector can see the partial order that the candidates up to that point have induced, and that the goal is to select, once again online, a maximal element (there may be more than one) of the underlying poset.

This topic was introduced in Stadje (1980), and Russian mathematicians covered it in a number of publications that were well-reviewed in Gnedin (1992). Preater (1999), Garrod and Morris (2013), Freij and Wästlund (2010), Georgiou et al. (2008), and Kumar et al. (2008) all considered efficient universal algorithms for families of posets whose

structure is unknown to the selector before to the search (2011). Ka'zmierczak (2013), Tkocz and Tkocz, and Ka'zmierczak identified the best methods for basic non-linear posets. In Garrod et al. (2012), a poset secretary dilemma was also taken into account, where each candidate has a twin who is equally qualified. Morayne discovered an ideal technique for the posets whose Hasse diagrams are full binary trees of a specified length (1998). In the poset variation of the secretary issue, other model assumptions may be made. Posets, for instance, have sides, therefore it seems sense to suppose that the selector can determine which side a specific element originates from. The full binary tree model used in Morayne is enhanced in this study (1998). In other words, in the original design, the selector can only view the poset caused by the components that have already arrived, without knowing which side of the tree the seen elements originated from. However, if we assume that the items on the left and right are black and that the selector can see these colors, then this information can be given. In reality, a model with additional color is an option. We suppose that the full subtrees in our underlying complete binary tree are colored differently starting at a certain level down (see Fig. 1 where four different colors are used below level four where we count the levels from the leaves). This type of colored complete binary tree will be referred to as CCBTk n, where n is the tree's height and k is the number of uncolored levels, or just

ISSN NO: 9726-001X

Volume 11 Issue 04 2023

JOURNAL OF CURRENT SCIENCE

CCBT. It doesn't matter whether the element appears after or before x if, during a search, an element x from the non-colored (upper) part of the tree appears, it will take the color of the first colored element related to x from the current permutation. (Fig. 2 illustrates the first seven consecutive observations of the selector for a permutation.

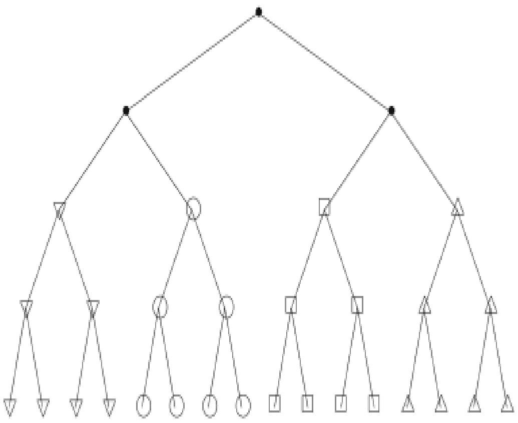


Fig. 1 $CCBT_5^2$

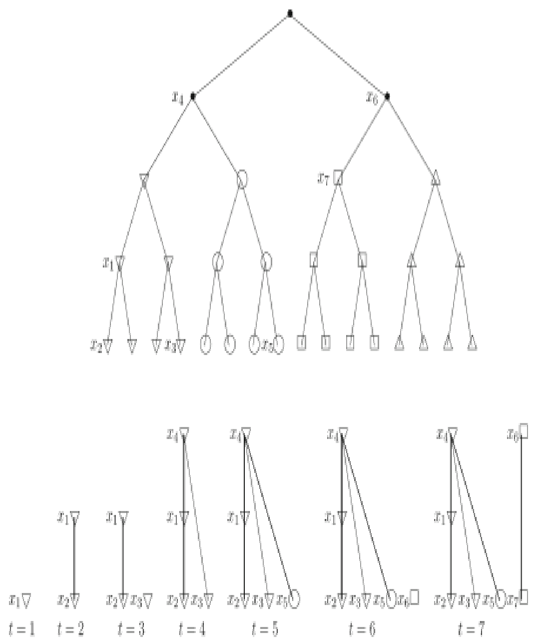


Fig. 2 Example of consecutive observations; note that the color ∇ of x_4 has been inherited from x_1 and the color $\Box$ of x_6 has been inherited from x_7 (because the next colored element related to x_6 in this permutation is x_7, despite the fact that x_7 appeared after x_6)

tation $((x_1, \nabla), (x_2, \nabla), (x_3, \nabla), (x_4, \nabla), (x_5, \circ), (x_6, \Box), (x_7, \Box)\ldots)$; note that the color $c_4 = \nabla$ of x_4 has been inherited from x_1 and the color $c_6 = \Box$ of x_6 has been inherited from x_7 (because the next colored element related to x_6 in this permutation is x_7, despite the fact that x_7 appeared after x_6); note also that at $t = 5$ x_4 is already identified as lying in the originally uncolored part because there are more than one colors below x_4).

In this note we limit ourselves to an informal treatment, referring the reader to, e.g. Morayne (1998) or Preater (1999) for further details. We hope the following description will be sufficient to follow the argument given and to enable the reader to add the formalism lacked.

We will refer to posets whose Hasse diagrams are trees simply as *trees*. We will also call complete binary trees *CBT* and complete binary trees of height n CBT_n.

Let $N = 2^n - 1$. The elementary events of our probability space are permutations $x = (x_1, x_2, \ldots, x_N)$ of the vertices of our *CCBT*; each such permutation has uniquely assigned sequence of colors $c(x) = (c_1, c_2, \ldots, c_N)$: c_i is the color of the vertex x_i if it is colored in our *CCBT*, or, if it is in the uncolored part, c_i is the color of the first colored element in the permutation x that is in the *CCBT* below x_i.

We deal with a stochastic process whose values are colored and labeled posets Π_t isomorphic to subposets of our *CCBT* induced by the first t elements $x_1, \ldots, x_t$ of x where vertices are labelled with the times they arrived at and have colors from $c(x)$.

We are looking for a stopping time $\tau : x \to \tau(x) \in \{1, \ldots, n\}$ such that the vertex $x_{\tau(x)}$ is equal to the root $\mathbf{1}$ of our *CCBT* with the maximal possible probability. The

ISSN NO: 9726-001X

Volume 11 Issue 04 2023

JOURNAL OF CURRENT SCIENCE

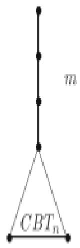


Fig. 3 $CBTA_n^m$

decision of selection is based only on the structure of Π_t and the information about colors of the elements of Π_t as described above. In other words the value t of $\tau(x)$ must be determinable only by what has happened by $t = \tau(x)$ (this exactly means that τ is a stopping time).

More formally let $\Omega = S_n$ (the family of all permutations of $1, \ldots, n$) and $\mathcal{F}_t$ be the σ-algebra of events that depend only of the first t elements of a permutation (the atoms of $\mathcal{F}_t$ are sets $A_{i_1,\ldots,i_t} = \{\pi : \pi \in S_n$ and $\pi_1 = i_1, \ldots, \pi_t = i_t\}$. A stopping time $\tau : \Omega \to \{1, \ldots, n\}$ is a random variable such that $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ ($\tau(i)$ depends only on what happened till time i). Let for $\pi \in \Omega$ $X_t(\pi) = 1$ if $\pi(t) = 1$ and $X_t(\pi) = 0$ otherwise. The selectors aim is to find a stopping time τ^* such that $P[X_{\tau^*} = 1] \geq P[X_\tau = 1]$ for all stopping times τ.

Let Y be a poset whose Hasse diagram consists of a chain of length $m - 1$ and a complete binary tree CBT_n under this chain (see Fig. 3). We will call such a poset a *complete binary tree with antenna*, $CBTA_n^m$ for short or simply *CBTA*.

The paper is organized as follows. Section 2 contains some combinatorial facts about counting embeddings of a tree into a tree. They will be necessary for estimating probabilities of success conditioned by the fact that the selector sees a specific structure at a given moment. In Sect. 3 we will find the strategy for $CCBT_n^l$ that will be near optimal in the following sense: for all multicolor structures and asymptotically almost all monochromatic ones the selector's decisions are optimal. For other monochromatic structures the strategy is optimal asymptotically.

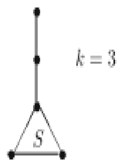
2 Embeddings of non-linear trees into *CBT* and *CB*

Let T be any tree. Let $l(T)$ be the number of leaves c
trees. Let S be a subset of T_1 such that S and T_2 are isc
S an *embedding* of T_2 into T_1. Let us call S a *good em*
of T_1 and a *bad embedding* if it does not contain the ro

Let $A_T^{m,n}$, $B_T^{m,n}$, $C_T^{m,n}$ be the number of good, bad, all
respectively. Let A_T^n, B_T^n, C_T^n be th number of good, l
CBT_n, respectively.

Throughout this section we establish several facts
mention that such counting problems stemming from tl
attracted independent attention and were considered i
2006), Kuchta et al. (2005, 2009) and Georgiou (2005

Fig. 4 S'



Let $k \in \{1, 2, \ldots\}$. Let S' be any tree whose first k biggest elements form a chain and the k'th element has more than one child. (see Fig. 4). Let S be the subset of S' which consists of all elements from S' except the first $k - 1$ ones. Let s be the height of S.

We will use the following well known elementary fact about the convergence of a sequence of series to a series (which is a discrete version and a consequence of Lebesgue's bounded convergence theorem). We will not prove it here.

Lemma 2.1 *Let $i_0 \in \mathbf{N}$. Let $0 \leq u_{i,n} \leq w_i$ for $i \geq i_0$ and $\sum_{i=0}^{\infty} w_i < \infty$. Then if $\lim_{n\to\infty} u_{i,n} = v_i$ then $\sum_{i=0}^{\infty} u_{i,n} \to \sum_{i=0}^{\infty} v_i$.*

We will also use the following technical lemma:

Lemma 2.2 $\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \left(\binom{c+i}{d} - \binom{c+i}{d-1} \right) = \binom{c}{d}$; *$c, d \in \mathbb{N}$ (we use the convention $\binom{c}{d} = 0$ for $c < d$).*

Proof Let

$$\begin{aligned} V &= \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \left(\binom{c+i}{d} - \binom{c+i}{d-1} \right) \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \left(\binom{c+i}{d} + \binom{c+i}{d-1} \right) - 2 \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \binom{c+i}{d-1} \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \binom{c+i+1}{d} - 2 \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \binom{c+i}{d-1} \\ &= \sum_{i=1}^{\infty} \frac{1}{2^{i}} \binom{c+i}{d} - \sum_{i=0}^{\infty} \frac{1}{2^{i}} \binom{c+i}{d-1} = 2V - \binom{c}{d}. \end{aligned}$$

Thus we get $V = \binom{c}{d}$. □

Now we will prove a series of lemmas comparing the numbers of particular embeddings into *CBT* and *CBTA*.

Lemma 2.3 $A_S^{n+1} \geq 2^{l(S)} A_S^n$.

Proof The proof goes along the same lines as the proof of Propostion 2.1 in Kubicki et al. (2003). □

ISSN NO: 9726-001X

Volume 11 Issue 04 2023

JOURNAL OF CURRENT SCIENCE

Lemma 2.4 $\lim_{n\to\infty} A_S^{n+1}/A_S^n = 2^{l(S)}$.

Proof From Kubicki et al. (2003) we know that $\lim_{n\to\infty} \frac{A_S^n}{B_S^n} = 2^{l(S)-1} - 1$ an $\lim_{n\to\infty} \frac{B_S^{n+1}}{B_S^n} = 2^{l(S)}$. Thus

$$\lim_{n\to\infty} \frac{A_S^{n+1}}{A_S^n} = \frac{\frac{A_S^{n+1}}{B_S^{n+1}}}{\frac{A_S^n}{B_S^n}} \cdot \frac{B_S^{n+1}}{B_S^n} = \frac{2^{l(S)-1}-1}{2^{l(S)-1}-1} \cdot 2^{l(S)} = 2^{l(S)}.$$

Let a_i be the number of embeddings of S into CBT_n such that the maximal elemer of S is on level i (the leaves of CBT_n are on level 1). Of course, $\frac{a_{n+1}}{a_n} = \frac{A_S^{n+1}}{2A_S^n}$. Le $2k = m + y$ for some $y \in \{1, 2, \ldots\}$. Let s be the height of S.

Lemma 2.5 *If $l(S') > 2$ and $2k > m \geq k$, then $A_{S'}^{m,n} > B_{S'}^{m,n}$.*

Proof Note that

$$A_{S'}^{m,n} = \sum_{i=s}^{n-1} \binom{n+m-i-2}{k-2} \cdot a_i + \binom{m-1}{k-1} \cdot a_n$$

and

$$B_{S'}^{m,n} = \sum_{i=s}^{n-1} \binom{n+m-i-2}{k-1} \cdot a_i + \binom{m-1}{k} \cdot a_n.$$

The inequality $A_{S'}^{m,n} > B_{S'}^{m,n}$ is equivalent to the inequality:

$$\sum_{i=s}^{n-1} \left(\binom{n+m-i-2}{k-1} - \binom{n+m-i-2}{k-2} \right) \cdot a_i < \left(\binom{m-1}{k-1} - \binom{m-1}{k} \right) \cdot a_n$$

which can be written as

$$\sum_{i=0}^{n-1-s} \left(\binom{n+m-i-2-s}{k-1} - \binom{n+m-i-2-s}{k-2} \right) \cdot a_{i+s} < \left(\binom{m-1}{k-1} - \binom{m-1}{k} \right) \cdot a_n.$$

Changing the order of summation we obtain the inequality

$$\sum_{i=0}^{n-1-s} \left(\binom{m-1+i}{k-1} - \binom{m-1+i}{k-2} \right) \cdot a_{n-1-i} < \left(\binom{m-1}{k-1} - \binom{m-1}{k} \right) \cdot a_n,$$

and replacing m by $2k - y$, dividing both sides by a_n and using $\binom{2k-y-1}{k-1} - \binom{2k-y-1}{k} = \binom{2k-y-1}{k-1} \frac{y}{k}$ we obtain

$$\sum_{i=0}^{n-1-s} \left(\binom{2k-y-1+i}{k-1} - \binom{2k-y-1+i}{k-2} \right) \cdot \frac{a_{n-1-i}}{a_n} < \binom{2k-y-1}{k-1} \frac{y}{k}. \tag{1}$$

Now removing from the left -hand side the terms lower than 0 and applying $\frac{a_{n-1-i}}{a_n} < \frac{1}{4^{i+1}}$ we get the following stronger inequality

$$\sum_{i=y-1}^{n-1-s} \left(\binom{2k-y-1+i}{k-1} - \binom{2k-y-1+i}{k-2} \right) \cdot \frac{1}{4^{i+1}} < \binom{2k-y-1}{k-1} \frac{y}{k}.$$

Now we will show that

$$\sum_{i=y-1}^{\infty} \left(\binom{2k-y-1+i}{k-1} - \binom{2k-y-1+i}{k-2} \right) \cdot \frac{1}{4^{i+1}} < \binom{2k-y-1}{k-1} \frac{y}{k},$$

or, equivalently,

$$\sum_{i=0}^{\infty} \left(\binom{2k-2+i}{k-1} - \binom{2k-2+i}{k-2} \right) \cdot \frac{1}{4^i} < 4^y \binom{2k-y-1}{k-1} \frac{y}{k}.$$

It is easy to show that the right-hand side of the inequality is minimal for $y = 1$. So it is enough to show that

$$\sum_{i=0}^{\infty} \left(\binom{2k-2+i}{k-1} - \binom{2k-2+i}{k-2} \right) \cdot \frac{1}{4^i} < 4 \binom{2k-2}{k-1} \frac{1}{k},$$

which is equivalent to

$$\sum_{i=0}^{\infty} \frac{(2k-2+1)(2k-2+2)\cdots(2k-2+i)(i+1)}{(k+1)(k+2)\cdots(k+i)4^{i+1}} < 1.$$

But $\frac{2k-1+c}{k+1+c} < 2$ for every $c \geq 0$. Thus the conclusion follows from the equality $\sum_{i=0}^{\infty} \frac{i+1}{2^{i+1}} = 2$. □

Lemma 2.6 *If $m < k$ and $l(S') \geq 2$, then $A_{S'}^{m,n} > B_{S'}^{m,n}$.*

Proof Note that $m < k$ means that $y = k + z$ for some $z \in \{1, 2, \ldots\}$. From the proof of Lemma 2.5 we know that the inequality $A_{S'}^{m,n} > B_{S'}^{m,n}$ is equivalent to inequality

JOURNAL OF CURRENT SCIENCE

(1) with the right-hand side equal to 0 (we assume $\binom{a}{b} = 0$ for $a < b$). So we have to prove that

$$\sum_{i=0}^{n-1-s}\left(\binom{2k-y-1+i}{k-1}-\binom{2k-y-1+i}{k-2}\right)\cdot\frac{a_{n-1-i}}{a_n}<0$$

which is equivalent to

$$\sum_{i=0}^{n-1-s}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot a_{n-1-i}<0.$$

Note that the first $k+z-2$ terms of the sum above are ≤ 0. We move them to the other side and we obtain the following inequality (note that if $k+z-2>n-1-s$ then our inequality is obvious so further we assume that $k+z-2\le n-1-s$).

$$\sum_{i=k+z-2}^{n-1-s}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot a_{n-1-i}$$
$$<-\sum_{i=0}^{k+z-3}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot a_{n-1-i}.$$

Now we shift a summation index, we divide both sides by $a_{n-k-z+2}$ and we obtain

$$\sum_{i=0}^{n-s-k-z+1}\left(\binom{2k-3+i}{k-1}-\binom{2k-3+i}{k-2}\right)\cdot\frac{a_{n-i-k-z+1}}{a_{n-k-z+2}}$$
$$<-\sum_{i=0}^{k+z-3}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot\frac{a_{n-1-i}}{a_{n-k-z+2}}.$$

Applying $\frac{a_{n-i}}{a_n}\le\frac{1}{2^i}$ and replacing the summation boundary by ∞ we get the follow stronger inequality:

$$\sum_{i=0}^{\infty}\left(\binom{2k-3+i}{k-1}-\binom{2k-3+i}{k-2}\right)\cdot\frac{1}{2^{i+1}}$$
$$\le-\sum_{i=0}^{k+z-3}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{k+z-3-i}.$$

Let L, R be the left-hand and the right-hand side of the inequality above, resp tively. Using $L<\infty$ (Lemma 2.2) we can write $-R$ as follows:

$$-R=\sum_{i=0}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{k+z-3-i}$$
$$-\sum_{i=k+z-2}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{k+z-3-i}$$
$$=2^{k+z-2}\sum_{i=0}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{-i-1}$$
$$-\sum_{i=0}^{\infty}\left(\binom{2k-3+i}{k-1}-\binom{2k-3+i}{k-2}\right)\cdot 2^{-1-i}$$
$$=2^{k+z-2}\sum_{i=0}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{-i-1}-L.$$

Hence (2) is equivalent to

$$L\le L-2^{k+z-2}\sum_{i=0}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{-i-1}.$$

Now using Lemma 2.2 we get

$$\sum_{i=0}^{\infty}\left(\binom{k-z-1+i}{k-1}-\binom{k-z-1+i}{k-2}\right)\cdot 2^{-i-1}=\binom{k-z-1}{k-1}=0$$

for $z>0$. □

Lemma 2.7 *For $l(S')=2$, if $y<k$ (i.e. $m>k$) then*

$$\lim_{n\to\infty}\frac{B^{m,n}_{S'}-A^{m,n}_{S'}}{a_n}>0$$

and if $y=k$ (i.e. $m=k$) then

$$\lim_{n\to\infty}\frac{B^{m,n}_{S'}-A^{m,n}_{S'}}{a_n}=0.$$

Proof From the proof of Lemma 2.5 (inequality (1)) the inequality $\frac{B^{m,n}_{S'}-A^{m,n}_{S'}}{a_n}>0$ takes the form:

$$\frac{B^{m,n}_{S'}-A^{m,n}_{S'}}{a_n}=\sum_{i=0}^{n-1-s}\left(\binom{2k-y-1+i}{k-1}-\binom{2k-y-1+i}{k-2}\right)\cdot\frac{a_{n-1-i}}{a_n}$$
$$-\binom{2k-y-1}{k-1}\frac{y}{k}.$$

ISSN NO: 9726-001X

Volume 11 Issue 04 2023

JOURNAL OF CURRENT SCIENCE

Now we use Lemma 2.1 for

$$u_{i,n} = \left(\left(\binom{2k-y-1+i}{k-1}\right) - \left(\binom{2k-\right.\right.$$

and

$$w_i = v_i = \left(\left(\binom{2k-y-1+i}{k-1}\right) - \left(\binom{2\right.\right.$$

We know that $u_{i,n} \to v_i$ (use $\frac{a_{n+1}}{a_n} = \frac{A_S^{n+1}}{2A_S^n}$ and L big enough, we have $u_{i,n} \leq v_i$ (Lemma 2.3).

Hence

$$\lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n} = V - \left(2\right.$$

where

$$V = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \left(\left(\binom{2k-y-1+i}{k-1}\right) - \right.$$

Now using Lemma 2.2 for $c = 2k - y - 1$ and $d = k - 1$ we obtain $V = \binom{c}{d} = \binom{2k-y-1}{k-1}$.

So the inequality $\lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n} > 0$ is equivalent to the inequality $k > y$. And, analogously, the equality $\lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n} = 0$ is equivalent to $k = y$. □

Lemma 2.8 *If $l(S') = 2$, then: if $y < k$ (i.e. $m > k$) then $\lim_{n\to\infty} B_{S'}^{m,n}/A_{S'}^{m,n} > 1$, and if $y = k$ (i.e. $m = k$) then $\lim_{n\to\infty} B_{S'}^{m,n}/A_{S'}^{m,n} = 1$.*

Proof First we will show that $0 < \lim_{n\to\infty} A_{S'}^{m,n}/a_n < \infty$.

From the proof of Lemma 2.5 we know that

$$\frac{A_{S'}^{m,n}}{a_n} = \sum_{i=s}^{n-1} \binom{n+m-i-2}{k-2} \cdot \frac{a_i}{a_n} + \binom{m-1}{k-1}$$

$$= \sum_{i=0}^{n-1-s} \binom{n+m-i-2-s}{k-2} \cdot \frac{a_{i+s}}{a_n} + \binom{m-1}{k-1}$$

$$= \sum_{i=0}^{n-1-s} \binom{m+i-1}{k-2} \cdot \frac{a_{n-1-i}}{a_n} + \binom{m-1}{k-1}$$

$$< \sum_{i=0}^{n-1-s} \binom{m+i-1}{k-2} \cdot \frac{1}{2^{i+1}} + \binom{m-1}{k-1}.$$

But

$$\sum_{i=0}^{\infty} \binom{m+i-1}{k-2} \cdot \frac{1}{2^{i+1}} < \frac{1}{(k-2)!} \sum_{i=0}^{\infty} (m+i-1)^{k-2} \cdot \frac{1}{2^{i+1}}$$

$$= \frac{1}{(k-2)!} \sum_{i=m-1}^{\infty} i^{k-2} \cdot \frac{1}{2^{i+2-m}}$$

$$= \frac{2^{m-2}}{(k-2)!} \sum_{i=m-1}^{\infty} \frac{i^{k-2}}{2^{i+1}} < \infty,$$

because $\sum_{i=0}^{\infty} \frac{i^c}{2^i} < \infty$ for any $c < \infty$. So by Lemma 2.1 $\lim_{n\to\infty} \frac{A_{S'}^{m,n}}{a_n}$ exists and $\lim_{n\to\infty} \frac{A_{S'}^{m,n}}{a_n} < \infty$. As $\binom{m-1}{k-1} \geq 1$ the inequality $0 < \lim_{n\to\infty} \frac{A_{S'}^{m,n}}{a_n}$ is obvious.

Now we get

$$\lim_{n\to\infty} \frac{B_{S'}^{m,n}}{A_{S'}^{m,n}} = 1 + \lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{A_{S'}^{m,n}}$$

$$= 1 + \lim_{n\to\infty} \frac{\frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n}}{\frac{A_{S'}^{m,n}}{a_n}}.$$

But $0 < \lim_{n\to\infty} \frac{A_{S'}^{m,n}}{a_n} < \infty$ and (by Lemma 2.7) if $y < k$ then $0 < \lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n} < \infty$ and $\lim_{n\to\infty} \frac{B_{S'}^{m,n} - A_{S'}^{m,n}}{a_n} = 0$ if $y = k$. □

3 Near optimal strategy

Recall that $CCBT_{\bar{n}}^m$ is a colored complete binary tree of height $\bar{n}$ with m non-colored levels where all complete binary subtrees below level m are colored with distinct colors. In this section we will define a stopping time τ_0 for our best choice problem for $CCBT_{\bar{n}}^m$. It is, in general, not optimal but *nearly* optimal in the sense that within the event of probability asymptotically equal to one it behaves in the optimal way and in the marginal situations, i.e. those of probability tending to zero, even if it is not optimal for some given fixed poset we deal with, it is either optimal for this poset from some $\bar{n}$ on or asymptotically this strategy gives us the same result as the optimal strategy.

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Let $n = \bar{n} - m + 1$. Let g be the number of embeddings of S' into $CBTA^m_n$ suc that the first k elements of S' are among the first $m = 2k$ elements of $CBTA^m_n$. Let be the number of remaining embeddings of S' into $CBTA^m_n$. Note that $g = \binom{2k}{k} A$ and $h \geq B^n_S \binom{2k}{k-1}$.

Now let us note that

$$P[[x_t = 1]|G_k] \leq \frac{1}{2}\frac{g}{g+h} + \frac{k-1}{2k}\frac{h}{g+h} = \frac{1}{2} - \frac{1}{2k}\frac{h}{g+h}$$

$$\leq \frac{1}{2} - \frac{1}{2k}\frac{B^n_S\binom{2k}{k-1}}{A^n_S\binom{2k}{k} + B^n_S\binom{2k}{k-1}}.$$

But we know that there exists some $c > 0$ such that from some n on (becau: $\lim_{n\to\infty} \frac{A^n_S}{B^n_S} = 2^{l(S)-1} - 1$). So we can write

$$P[[x_t = 1]|G_k] \leq \frac{1}{2} - \frac{c}{2k}.$$

On the other hand

$$P[[x_{\tau_S} = 1]|G_k \cap [x_t \neq 1]] = 1 - \frac{1}{2^{\bar{n}-1}}.$$

So our inequality follows from

$$\frac{1}{2} - \frac{c}{2k} \leq \left(\frac{1}{2} + \frac{c}{2k}\right)\cdot\left(1 - \frac{1}{2^{\bar{n}-1}}\right),$$

which is true for some n_0 and $\bar{n} > n_0$, because $\lim_{\bar{n}\to\infty} \frac{k}{2^{\bar{n}-1}} = 0$ and $c > 0$. □

Theorem 3.2 *Let $x_t = \max\{x_1, \ldots, x_t\}$ and $x_1, \ldots, x_t$ form a monochromatic non-linear order $S' \in \mathcal{S}'(k)$. If $2k < m$ then playing optimal strategy we should not stop.*

Proof As in the proof of Theorem 3.1 let G_k be an event such that $x_1, \ldots, x_t$ form S'. We want to show that

$$P[[x_t = 1]|G_k] < P[[x_{\tau_S} = 1]|G_k \cap [x_t \neq 1]] \cdot P[[x_t \neq 1]|G_k].$$

Note that $P[[x_t = 1]|G_k] \leq \frac{k}{m} \leq \frac{m-1}{2m}$ and $P[[x_{\tau_S} = 1]|G_k \cap [x_t \neq 1]] = \frac{2^{\bar{n}-1}-1}{2^{\bar{n}-1}}$. So we need to show that $m - 1 < (m+1)\frac{2^{\bar{n}-1}-1}{2^{\bar{n}-1}}$ which is obviously true. □

The theorems above justify our claim that τ_0 is near-optimal in the sense stated in the beginning of this section.

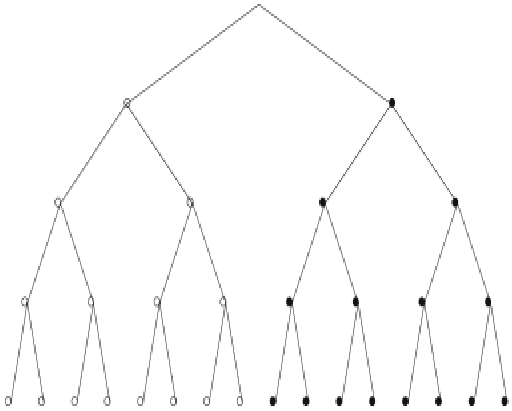

Fig. 5 $CCBT^1_5$

4 Optimal stopping time for two-colored complete binary tree $CCBT^1_{\bar{n}}$

For the case $CCBT^1_{\bar{n}}$, i.e. when a $CBT_{\bar{n}}$ is colored with only two colors (say the right-hand side is black and the left-hand side is white, see Fig. 5) we can find an optimal stopping time τ.

Let us define τ as the stopping time such that $\tau = t$ if and only if t is the first time such that $x_t = \max\{x_1, \ldots, x_t\}$ and one of the following situations occurs:

(1) $x_1, \ldots, x_t$ form a chain and $2t > \bar{n}$;
(2) $x_1, \ldots, x_t$ are colored with 2 different colors;
(3) $x_1, \ldots, x_t$ form a monochromatic non-linear order $S' \in \mathcal{S}'(k)$ and $k > 1$.

If none of these situations occurs then $\tau = 2^{\bar{n}} - 1$.

Note that this strategy is the near-optimal strategy from the previous section for the case of two colors.

Let us denote by $D_{i,t}$ the event when $\{x_1, \ldots, x_t\}$ form a monochromatic non-linear order $S' \in \mathcal{S}'(i)$ and $x_t = \max\{x_1, \ldots, x_t\}$. Let U be the order constructed from S' by removing from S' the maximal element.

Theorem 4.1 *The stopping time τ is optimal for $CCBT^1_{\bar{n}}$.*

Proof The optimality of τ in situations (1) and (2) was proved in the previous sections. Now we will show that for $D_{i,t}$ for $i > 1$ we should stop.

Let T be any non-linear order with one maximal element. Let A_T, B_T, C_T be the number of good, bad, all embeddings of T into CBT_n, respectively. Let A'_T, B'_T, C'_T be a number of good, bad, all embeddings of T into $CBTA^2_n$, respectively. From Morayne (1998) we know that $A_T > B_T$. We will show that $A'_{S'} > B'_{S'}$.

It is enough to notice that $A'_{S'} = C_U$, $B'_{S'} = C_{S'}$ and $A_{S'} = B_U$. Note also that the inequality $A_T > B_T$ is equivalent to each of the inequalities $C_T > 2B_T$ and $2A_T > C_T$ (because $C_T = A_T + B_T$). Now we can write

$$A'_{S'} = C_U > 2B_U = 2A_{S'} > C_{S'} = B'_{S'},$$

thus we should stop.

It remains to show that stopping for $D_{1,t}$ is not optimal. Assume that none of situations (1), (2) and (3) occurred before time t. Let **2** be the son of **1** which has the color of S'.

First note that

$$P[[x_t = \mathbf{1}]|D_{1,t}] = P[[x_t = \mathbf{2}]|D_{1,t}]$$

and

$$P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t = \mathbf{2}]] = 1.$$

We want to show that

$$P[[x_\tau = \mathbf{1}]|D_{1,t}] \le P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t \ne \mathbf{1}]] \cdot P[[x_t \ne \mathbf{1}]|D_{1,t}].$$

But

$$\begin{aligned}
&P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t \ne \mathbf{1}]] \cdot P[[x_t \ne \mathbf{1}]|D_{1,t}]\\
&= P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t \ne \mathbf{1}] \cap [x_t = \mathbf{2}]]\\
&\cdot P[[x_t = \mathbf{2}]|D_{1,t} \cap [x_t \ne \mathbf{1}]] \cdot P[[x_t \ne \mathbf{1}]|D_{1,t}]\\
&+ P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t \ne \mathbf{1}] \cap [x_t \ne \mathbf{2}]]\\
&\cdot P[[x_t \ne \mathbf{2}]|D_{1,t} \cap [x_t \ne \mathbf{1}]] \cdot P[[x_t \ne \mathbf{1}]|D_{1,t}]\\
&\ge P[[x_\tau = \mathbf{1}]|D_{1,t} \cap [x_t \ne \mathbf{1}] \cap [x_t = \mathbf{2}]]\\
&\cdot P[[x_t = \mathbf{2}]|D_{1,t} \cap [x_t \ne \mathbf{1}]] \cdot P[[x_t \ne \mathbf{1}]|D_{1,t}]\\
&= P[[x_t = \mathbf{2}] \cap [x_t \ne \mathbf{1}]|D_{1,t}] = P[[x_t = \mathbf{2}]|D_{1,t}] = P[[x_t = \mathbf{1}]|D_{1,t}].
\end{aligned}$$

It is interesting to compare the efficiency of optim
complete binary trees and the non-colored complete
The difference between these two cases appears
chromatic order $S' \in \mathcal{S}'(1)$ and the last element we
stopped earlier. In such situations in the case of two-
continue and in the case of non-colored complete bi
(1998)).

Thus in the first case we make a mistake $2A^{S'}_{\bar{n}-1}$ tim
times, where A^T_n, C^T_n is the number of good,all embe
respectively.

Let P_1, P_2 be the probabilities of making a mist
non-colored one, respectively, in the situations when
$P_{S'}$ be the probability of the event that at some time
and the decisions at the moment t in both cases are d

Because we know from Morayne (1998) that $2A^{S'}_n \ge C^{S'}_n$ we get

$$P_1 = \sum_{S' \in \mathcal{S}'(1)} P_{S'} \frac{A^{S'}_{\bar{n}-1}}{A^{S'}_{\bar{n}-1} + C^{S'}_{\bar{n}-1}} \ge \sum_{S' \in \mathcal{S}'(1)} P_{S'} \frac{C^{S'}_{\bar{n}-1}}{2(A^{S'}_{\bar{n}-1} + C^{S'}_{\bar{n}-1})} = \frac{1}{2} P_2.$$

So we can see that, rather surprisingly, a two-coloring of *CBT*, even in the (marginal) situations where the strategies differ, does not reduce the probability of mistake more than twice.

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